

# Boundedness of intrinsic square functions on the weighted weak Hardy spaces

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## Abstract

In this paper, by using the atomic decomposition theorem for weighted weak Hardy spaces, we will show the boundedness properties of intrinsic square functions including the Lusin area integral, Littlewood-Paley  $g$ -function and  $g_\lambda^*$ -function on these spaces.

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## 1 Introduction

Let  $\mathbb{R}_+^{n+1} = \mathbb{R}^n \times (0, \infty)$  and  $\varphi_t(x) = t^{-n}\varphi(x/t)$ . The classical square function (Lusin area integral) is a familiar object. If  $u(x, t) = P_t * f(x)$  is the Poisson integral of  $f$ , where  $P_t(x) = c_n \frac{t}{(t^2 + |x|^2)^{(n+1)/2}}$  denotes the Poisson kernel in  $\mathbb{R}_+^{n+1}$ . Then we define the classical square function (Lusin area integral)  $S(f)$  by

$$S(f)(x) = \left( \iint_{\Gamma(x)} |\nabla u(y, t)|^2 t^{1-n} dy dt \right)^{1/2},$$

where  $\Gamma(x)$  denotes the usual cone of aperture one:

$$\Gamma(x) = \{(y, t) \in \mathbb{R}_+^{n+1} : |x - y| < t\}$$

and

$$|\nabla u(y, t)| = \left| \frac{\partial u}{\partial t} \right|^2 + \sum_{j=1}^n \left| \frac{\partial u}{\partial y_j} \right|^2.$$

We can similarly define a cone of aperture  $\beta$  for any  $\beta > 0$ :

$$\Gamma_\beta(x) = \{(y, t) \in \mathbb{R}_+^{n+1} : |x - y| < \beta t\},$$

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and corresponding square function

$$S_\beta(f)(x) = \left( \iint_{\Gamma_\beta(x)} |\nabla u(y, t)|^2 t^{1-n} dy dt \right)^{1/2}.$$

The Littlewood-Paley  $g$ -function (could be viewed as a “zero-aperture” version of  $S(f)$ ) and the  $g_\lambda^*$ -function (could be viewed as an “infinite aperture” version of  $S(f)$ ) are defined respectively by

$$g(f)(x) = \left( \int_0^\infty |\nabla u(x, t)|^2 t dt \right)^{1/2}$$

and

$$g_\lambda^*(f)(x) = \left( \iint_{\mathbb{R}_+^{n+1}} \left( \frac{t}{t + |x - y|} \right)^{\lambda n} |\nabla u(y, t)|^2 t^{1-n} dy dt \right)^{1/2}.$$

The modern (real-variable) variant of  $S_\beta(f)$  can be defined in the following way. Let  $\psi \in C^\infty(\mathbb{R}^n)$  be real, radial, have support contained in  $\{x : |x| \leq 1\}$ , and  $\int_{\mathbb{R}^n} \psi(x) dx = 0$ . The continuous square function  $S_{\psi, \beta}(f)$  is defined by

$$S_{\psi, \beta}(f)(x) = \left( \iint_{\Gamma_\beta(x)} |f * \psi_t(y)|^2 \frac{dy dt}{t^{n+1}} \right)^{1/2}.$$

In 2007, Wilson [19] introduced a new square function called intrinsic square function which is universal in a sense (see also [20]). This function is independent of any particular kernel  $\psi$ , and it dominates pointwise all the above defined square functions. On the other hand, it is not essentially larger than any particular  $S_{\psi, \beta}(f)$ . For  $0 < \alpha \leq 1$ , let  $\mathcal{C}_\alpha$  be the family of functions  $\varphi$  defined on  $\mathbb{R}^n$  such that  $\varphi$  has support containing in  $\{x \in \mathbb{R}^n : |x| \leq 1\}$ ,  $\int_{\mathbb{R}^n} \varphi(x) dx = 0$ , and, for all  $x, x' \in \mathbb{R}^n$ ,

$$|\varphi(x) - \varphi(x')| \leq |x - x'|^\alpha.$$

For  $(y, t) \in \mathbb{R}_+^{n+1}$  and  $f \in L_{loc}^1(\mathbb{R}^n)$ , we set

$$A_\alpha(f)(y, t) = \sup_{\varphi \in \mathcal{C}_\alpha} |f * \varphi_t(y)|.$$

Then we define the intrinsic square function of  $f$  (of order  $\alpha$ ) by the formula

$$\mathcal{S}_\alpha(f)(x) = \left( \iint_{\Gamma(x)} \left( A_\alpha(f)(y, t) \right)^2 \frac{dy dt}{t^{n+1}} \right)^{1/2}.$$

We can also define varying-aperture versions of  $\mathcal{S}_\alpha(f)$  by the formula

$$\mathcal{S}_{\alpha, \beta}(f)(x) = \left( \iint_{\Gamma_\beta(x)} \left( A_\alpha(f)(y, t) \right)^2 \frac{dy dt}{t^{n+1}} \right)^{1/2}.$$

The intrinsic Littlewood-Paley  $g$ -function and the intrinsic  $g_\lambda^*$ -function will be defined respectively by

$$g_\alpha(f)(x) = \left( \int_0^\infty \left( A_\alpha(f)(x, t) \right)^2 \frac{dt}{t} \right)^{1/2}$$

and

$$g_{\lambda, \alpha}^*(f)(x) = \left( \iint_{\mathbb{R}_+^{n+1}} \left( \frac{t}{t + |x - y|} \right)^{\lambda n} \left( A_\alpha(f)(y, t) \right)^2 \frac{dy dt}{t^{n+1}} \right)^{1/2}.$$

In [20], Wilson proved the following result.

**Theorem A.** *Let  $w \in A_p$  (Muckenhoupt weight class),  $1 < p < \infty$  and  $0 < \alpha \leq 1$ . Then there exists a constant  $C > 0$  independent of  $f$  such that*

$$\|\mathcal{S}_\alpha(f)\|_{L_w^p} \leq C \|f\|_{L_w^p}.$$

Moreover, in [12], Lerner showed sharp  $L_w^p$  norm inequalities for the intrinsic square functions in terms of the  $A_p$  characteristic constant of  $w$  for all  $1 < p < \infty$ . In [10], Huang and Liu studied the boundedness of intrinsic square functions on the weighted Hardy spaces  $H_w^1(\mathbb{R}^n)$ . Furthermore, they obtained the intrinsic square function characterizations of  $H_w^1(\mathbb{R}^n)$ . Recently, in [17] and [18], we have established the strong and weak type estimates of intrinsic square functions on the weighted Hardy spaces  $H_w^p(\mathbb{R}^n)$  for  $n/(n + \alpha) \leq p < 1$ .

The main purpose of this paper is to investigate the mapping properties of intrinsic square functions on the weighted weak Hardy spaces  $WH_w^p(\mathbb{R}^n)$  (see Section 2 for the definition). We now present our main results as follows.

**Theorem 1.1.** *Let  $0 < \alpha \leq 1$ ,  $n/(n + \alpha) < p \leq 1$  and  $w \in A_{p(1 + \frac{\alpha}{n})}$ . Then there exists a constant  $C > 0$  independent of  $f$  such that*

$$\|\mathcal{S}_\alpha(f)\|_{WL_w^p} \leq C \|f\|_{WH_w^p}.$$

**Theorem 1.2.** *Let  $0 < \alpha \leq 1$ ,  $n/(n + \alpha) < p \leq 1$  and  $w \in A_{p(1 + \frac{\alpha}{n})}$ . Suppose that  $\lambda > (3n + 2\alpha)/n$ , then there exists a constant  $C > 0$  independent of  $f$  such that*

$$\|g_{\lambda, \alpha}^*(f)\|_{WL_w^p} \leq C \|f\|_{WH_w^p}.$$

In [19], Wilson also showed that for any  $0 < \alpha \leq 1$ , the functions  $g_\alpha(f)(x)$  and  $\mathcal{S}_\alpha(f)(x)$  are pointwise comparable. Thus, as a direct consequence of Theorem 1.1, we obtain the following

**Corollary 1.3.** *Let  $0 < \alpha \leq 1$ ,  $n/(n + \alpha) < p \leq 1$  and  $w \in A_{p(1 + \frac{\alpha}{n})}$ . Then there exists a constant  $C > 0$  independent of  $f$  such that*

$$\|g_\alpha(f)\|_{WL_w^p} \leq C \|f\|_{WH_w^p}.$$

## 2 Notations and preliminaries

### 2.1 $A_p$ weights

The definition of  $A_p$  class was first used by Muckenhoupt [14], Hunt, Muckenhoupt and Wheeden [11], and Coifman and Fefferman [1] in the study of weighted  $L^p$  boundedness of Hardy-Littlewood maximal functions and singular integrals. Let  $w$  be a nonnegative, locally integrable function defined on  $\mathbb{R}^n$ ; all cubes are assumed to have their sides parallel to the coordinate axes. We say that  $w \in A_p$ ,  $1 < p < \infty$ , if

$$\left( \frac{1}{|Q|} \int_Q w(x) dx \right) \left( \frac{1}{|Q|} \int_Q w(x)^{-1/(p-1)} dx \right)^{p-1} \leq C \quad \text{for every cube } Q \subseteq \mathbb{R}^n,$$

where  $C$  is a positive constant which is independent of the choice of  $Q$ .

For the case  $p = 1$ ,  $w \in A_1$ , if

$$\frac{1}{|Q|} \int_Q w(x) dx \leq C \cdot \operatorname{ess\,inf}_{x \in Q} w(x) \quad \text{for every cube } Q \subseteq \mathbb{R}^n.$$

A weight function  $w \in A_\infty$  if it satisfies the  $A_p$  condition for some  $1 < p < \infty$ . It is well known that if  $w \in A_p$  with  $1 < p < \infty$ , then  $w \in A_r$  for all  $r > p$ , and  $w \in A_q$  for some  $1 < q < p$ . We thus write  $q_w \equiv \inf\{q > 1 : w \in A_q\}$  to denote the critical index of  $w$ .

Given a cube  $Q$  and  $\lambda > 0$ ,  $\lambda Q$  denotes the cube with the same center as  $Q$  whose side length is  $\lambda$  times that of  $Q$ .  $Q = Q(x_0, r)$  denotes the cube centered at  $x_0$  with side length  $r$ . For a weight function  $w$  and a measurable set  $E$ , we denote the Lebesgue measure of  $E$  by  $|E|$  and set the weighted measure  $w(E) = \int_E w(x) dx$ .

We give the following results that will be used in the sequel.

**Lemma 2.1** ([9]). *Let  $w \in A_q$  with  $q \geq 1$ . Then, for any cube  $Q$ , there exists an absolute constant  $C > 0$  such that*

$$w(2Q) \leq C w(Q).$$

*In general, for any  $\lambda > 1$ , we have*

$$w(\lambda Q) \leq C \cdot \lambda^{nq} w(Q),$$

*where  $C$  does not depend on  $Q$  nor on  $\lambda$ .*

**Lemma 2.2** ([9]). *Let  $w \in A_q$  with  $q > 1$ . Then, for all  $r > 0$ , there exists a constant  $C > 0$  independent of  $r$  such that*

$$\int_{|x| \geq r} \frac{w(x)}{|x|^{nq}} dx \leq C \cdot r^{-nq} w(Q(0, 2r)).$$

Given a weight function  $w$  on  $\mathbb{R}^n$ , for  $0 < p < \infty$ , we denote by  $L_w^p(\mathbb{R}^n)$  the weighted space of all functions  $f$  satisfying

$$\|f\|_{L_w^p} = \left( \int_{\mathbb{R}^n} |f(x)|^p w(x) dx \right)^{1/p} < \infty.$$

When  $p = \infty$ ,  $L_w^\infty(\mathbb{R}^n)$  will be taken to mean  $L^\infty(\mathbb{R}^n)$ , and

$$\|f\|_{L_w^\infty} = \|f\|_{L^\infty} = \operatorname{ess\,sup}_{x \in \mathbb{R}^n} |f(x)|.$$

We also denote by  $WL_w^p(\mathbb{R}^n)$  the weighted weak  $L^p$  space which is formed by all measurable functions  $f$  satisfying

$$\|f\|_{WL_w^p} = \sup_{\lambda > 0} \lambda \cdot w(\{x \in \mathbb{R}^n : |f(x)| > \lambda\})^{1/p} < \infty.$$

## 2.2 Weighted weak Hardy spaces

Let us now turn to the weighted weak Hardy spaces. The (unweighted) weak  $H^p$  spaces have first appeared in the work of Fefferman, Rivière and Sagher [7]. The atomic decomposition theory of weak  $H^1$  space on  $\mathbb{R}^n$  was given by Fefferman and Soria in [8]. Later, Liu [13] established the weak  $H^p$  spaces on homogeneous groups. For the boundedness properties of some operators on weak Hardy spaces, we refer the reader to [2–6] and [16]. In 2000, Quek and Yang [15] introduced the weighted weak Hardy spaces  $WH_w^p(\mathbb{R}^n)$  and established their atomic decompositions. Moreover, by using the atomic decomposition theory of  $WH_w^p(\mathbb{R}^n)$ , Quek and Yang [15] also obtained the boundedness of Calderón-Zygmund type operators on these weighted spaces.

We write  $\mathcal{S}(\mathbb{R}^n)$  to denote the Schwartz space of all rapidly decreasing smooth functions and  $\mathcal{S}'(\mathbb{R}^n)$  to denote the space of all tempered distributions, i.e., the topological dual of  $\mathcal{S}(\mathbb{R}^n)$ . Let  $w \in A_\infty$ ,  $0 < p \leq 1$  and  $N = [n(q_w/p - 1)]$ . Define

$$\mathcal{A}_{N,w} = \left\{ \varphi \in \mathcal{S}(\mathbb{R}^n) : \sup_{x \in \mathbb{R}^n} \sup_{|\alpha| \leq N+1} (1 + |x|)^{N+n+1} |D^\alpha \varphi(x)| \leq 1 \right\},$$

where  $\alpha = (\alpha_1, \dots, \alpha_n) \in (\mathbb{N} \cup \{0\})^n$ ,  $|\alpha| = \alpha_1 + \dots + \alpha_n$ , and

$$D^\alpha \varphi = \frac{\partial^{|\alpha|} \varphi}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}.$$

For  $f \in \mathcal{S}'(\mathbb{R}^n)$ , the grand maximal function of  $f$  is defined by

$$G_w f(x) = \sup_{\varphi \in \mathcal{A}_{N,w}} \sup_{|y-x| < t} |(\varphi_t * f)(y)|.$$

Then we can define the weighted weak Hardy space  $WH_w^p(\mathbb{R}^n)$  by  $WH_w^p(\mathbb{R}^n) = \{f \in \mathcal{S}'(\mathbb{R}^n) : G_w f \in WL_w^p(\mathbb{R}^n)\}$ . Moreover, we set  $\|f\|_{WH_w^p} = \|G_w f\|_{WL_w^p}$ .

**Theorem 2.3** ([15]). *Let  $0 < p \leq 1$  and  $w \in A_\infty$ . For every  $f \in WH_w^p(\mathbb{R}^n)$ , there exists a sequence of bounded measurable functions  $\{f_k\}_{k=-\infty}^\infty$  such that*

- (i)  $f = \sum_{k=-\infty}^\infty f_k$  in the sense of distributions.
  - (ii) Each  $f_k$  can be further decomposed into  $f_k = \sum_i b_i^k$ , where  $\{b_i^k\}$  satisfies
    - (a) Each  $b_i^k$  is supported in a cube  $Q_i^k$  with  $\sum_i w(Q_i^k) \leq c2^{-kp}$ , and  $\sum_i \chi_{Q_i^k}(x) \leq c$ . Here  $\chi_E$  denotes the characteristic function of the set  $E$  and  $c \sim \|f\|_{WH_w^p}^p$ ;
    - (b)  $\|b_i^k\|_{L^\infty} \leq C2^k$ , where  $C > 0$  is independent of  $i$  and  $k$ ;
    - (c)  $\int_{\mathbb{R}^n} b_i^k(x) x^\alpha dx = 0$  for every multi-index  $\alpha$  with  $|\alpha| \leq [n(q_w/p - 1)]$ .
- Conversely, if  $f \in \mathcal{S}'(\mathbb{R}^n)$  has a decomposition satisfying (i) and (ii), then  $f \in WH_w^p(\mathbb{R}^n)$ . Moreover, we have  $\|f\|_{WH_w^p}^p \sim c$ .

Throughout this article  $C$  always denote a positive constant independent of the main parameters involved, but it may be different from line to line.

### 3 Proof of Theorem 1.1

Before proving our main theorem in this section, let us first establish the following lemma.

**Lemma 3.1.** *Let  $0 < \alpha \leq 1$ . Then for any given function  $b \in L^\infty(\mathbb{R}^n)$  with support contained in  $Q = Q(x_0, r)$ , and  $\int_{\mathbb{R}^n} b(x) dx = 0$ , we have*

$$\mathcal{S}_\alpha(b)(x) \leq C \cdot \|b\|_{L^\infty} \frac{r^{n+\alpha}}{|x - x_0|^{n+\alpha}}, \quad \text{whenever } |x - x_0| > \sqrt{n}r.$$

*Proof.* For any  $\varphi \in \mathcal{C}_\alpha$ ,  $0 < \alpha \leq 1$ , by the vanishing moment condition of  $b$ , we have that for any  $(y, t) \in \Gamma(x)$ ,

$$\begin{aligned} |(b * \varphi_t)(y)| &= \left| \int_Q (\varphi_t(y - z) - \varphi_t(y - x_0)) b(z) dz \right| \\ &\leq \int_Q \frac{|z - x_0|^\alpha}{t^{n+\alpha}} |b(z)| dz \\ &\leq C \cdot \|b\|_{L^\infty} \frac{r^{n+\alpha}}{t^{n+\alpha}}. \end{aligned} \tag{1}$$

For any  $z \in Q$ , we have  $|z - x_0| \leq \frac{\sqrt{n}}{2}r < \frac{|x - x_0|}{2}$ . Furthermore, we observe that  $\text{supp } \varphi \subseteq \{x \in \mathbb{R}^n : |x| \leq 1\}$ , then for any  $(y, t) \in \Gamma(x)$ , by a direct computation, we can easily see that

$$2t \geq |x - y| + |y - z| \geq |x - z| \geq |x - x_0| - |z - x_0| \geq \frac{|x - x_0|}{2}. \tag{2}$$

Thus, for any point  $x$  with  $|x - x_0| > \sqrt{n}r$ , it follows from the inequalities (1)

and (2) that

$$\begin{aligned}
|\mathcal{S}_\alpha(b)(x)| &= \left( \iint_{\Gamma(x)} \left( \sup_{\varphi \in \mathcal{C}_\alpha} |(\varphi_t * b)(y)| \right)^2 \frac{dy dt}{t^{n+1}} \right)^{1/2} \\
&\leq C \cdot \|b\|_{L^\infty} r^{n+\alpha} \left( \int_{\frac{|x-x_0|}{4}}^\infty \int_{|y-x|<t} \frac{dy dt}{t^{2(n+\alpha)+n+1}} \right)^{1/2} \\
&\leq C \cdot \|b\|_{L^\infty} r^{n+\alpha} \left( \int_{\frac{|x-x_0|}{4}}^\infty \frac{dt}{t^{2(n+\alpha)+1}} \right)^{1/2} \\
&\leq C \cdot \|b\|_{L^\infty} \frac{r^{n+\alpha}}{|x-x_0|^{n+\alpha}}.
\end{aligned}$$

We are done.  $\square$

We are now in a position to give the proof of Theorem 1.1.

*Proof of Theorem 1.1.* For any given  $\lambda > 0$ , we may choose  $k_0 \in \mathbb{Z}$  such that  $2^{k_0} \leq \lambda < 2^{k_0+1}$ . For every  $f \in WH_w^p(\mathbb{R}^n)$ , then by Theorem 2.3, we can write

$$f = \sum_{k=-\infty}^{\infty} f_k = \sum_{k=-\infty}^{k_0} f_k + \sum_{k=k_0+1}^{\infty} f_k = F_1 + F_2,$$

where  $F_1 = \sum_{k=-\infty}^{k_0} f_k = \sum_{k=-\infty}^{k_0} \sum_i b_i^k$ ,  $F_2 = \sum_{k=k_0+1}^{\infty} f_k = \sum_{k=k_0+1}^{\infty} \sum_i b_i^k$  and  $\{b_i^k\}$  satisfies (a)–(c) in Theorem 2.3. Then we have

$$\begin{aligned}
&\lambda^p \cdot w(\{x \in \mathbb{R}^n : |\mathcal{S}_\alpha(f)(x)| > \lambda\}) \\
&\leq \lambda^p \cdot w(\{x \in \mathbb{R}^n : |\mathcal{S}_\alpha(F_1)(x)| > \lambda/2\}) + \lambda^p \cdot w(\{x \in \mathbb{R}^n : |\mathcal{S}_\alpha(F_2)(x)| > \lambda/2\}) \\
&= I_1 + I_2.
\end{aligned}$$

First we claim that the following inequality holds:

$$\|F_1\|_{L_w^2} \leq C \cdot \lambda^{1-p/2} \|f\|_{WH_w^p}^{p/2}. \quad (3)$$

In fact, since  $\text{supp } b_i^k \subseteq Q_i^k = Q(x_i^k, r_i^k)$  and  $\|b_i^k\|_{L^\infty} \leq C2^k$  by Theorem 2.3, then it follows from Minkowski's integral inequality that

$$\begin{aligned}
\|F_1\|_{L_w^2} &\leq \sum_{k=-\infty}^{k_0} \sum_i \|b_i^k\|_{L_w^2} \\
&\leq \sum_{k=-\infty}^{k_0} \sum_i \|b_i^k\|_{L^\infty} w(Q_i^k)^{1/2}.
\end{aligned}$$

For each  $k \in \mathbb{Z}$ , by using the bounded overlapping property of the cubes  $\{Q_i^k\}$  and the fact that  $1 - p/2 > 0$ , we thus obtain

$$\begin{aligned}
\|F_1\|_{L_w^2} &\leq C \sum_{k=-\infty}^{k_0} 2^k \left( \sum_i w(Q_i^k) \right)^{1/2} \\
&\leq C \sum_{k=-\infty}^{k_0} 2^{k(1-p/2)} \|f\|_{WH_w^p}^{p/2} \\
&\leq C \sum_{k=-\infty}^{k_0} 2^{(k-k_0)(1-p/2)} \cdot \lambda^{1-p/2} \|f\|_{WH_w^p}^{p/2} \\
&\leq C \cdot \lambda^{1-p/2} \|f\|_{WH_w^p}^{p/2}.
\end{aligned}$$

Since  $w \in A_{p(1+\frac{\alpha}{n})}$  and  $1 < p(1+\frac{\alpha}{n}) \leq 1 + \frac{\alpha}{n} \leq 2$ , then  $w \in A_2$ . Hence, it follows from Chebyshev's inequality and Theorem A that

$$\begin{aligned}
I_1 &\leq \lambda^p \cdot \frac{4}{\lambda^2} \|\mathcal{S}_\alpha(F_1)\|_{L_w^2}^2 \\
&\leq C \cdot \lambda^{p-2} \|F_1\|_{L_w^2}^2 \\
&\leq C \|f\|_{WH_w^p}^p.
\end{aligned}$$

Now we turn our attention to the estimate of  $I_2$ . If we set

$$A_{k_0} = \bigcup_{k=k_0+1}^{\infty} \bigcup_i \widetilde{Q}_i^k,$$

where  $\widetilde{Q}_i^k = Q(x_i^k, \tau^{(k-k_0)/(n+\alpha)}(2\sqrt{n})r_i^k)$  and  $\tau$  is a fixed positive number such that  $1 < \tau < 2$ . Thus, we can further decompose  $I_2$  as

$$\begin{aligned}
I_2 &\leq \lambda^p \cdot w(\{x \in A_{k_0} : |\mathcal{S}_\alpha(F_2)(x)| > \lambda/2\}) + \lambda^p \cdot w(\{x \in (A_{k_0})^c : |\mathcal{S}_\alpha(F_2)(x)| > \lambda/2\}) \\
&= I'_2 + I''_2.
\end{aligned}$$

Since  $w \in A_{p(1+\frac{\alpha}{n})}$ , then by Lemma 2.1, we can get

$$\begin{aligned}
I'_2 &\leq \lambda^p \sum_{k=k_0+1}^{\infty} \sum_i w(\widetilde{Q}_i^k) \\
&\leq C \cdot \lambda^p \sum_{k=k_0+1}^{\infty} \tau^{(k-k_0)p} \sum_i w(Q_i^k) \\
&\leq C \|f\|_{WH_w^p}^p \sum_{k=k_0+1}^{\infty} \left(\frac{\tau}{2}\right)^{(k-k_0)p} \\
&\leq C \|f\|_{WH_w^p}^p.
\end{aligned}$$



On the other hand, an application of Chebyshev's inequality gives us that

$$\begin{aligned} I_2'' &\leq 2^p \int_{(A_{k_0})^c} |\mathcal{S}_\alpha(F_2)(x)|^p w(x) dx \\ &\leq 2^p \sum_{k=k_0+1}^{\infty} \sum_i \int_{(\widetilde{Q}_i^k)^c} |\mathcal{S}_\alpha(b_i^k)(x)|^p w(x) dx. \end{aligned}$$

When  $x \in (\widetilde{Q}_i^k)^c$ , then a direct calculation shows that  $|x - x_i^k| \geq \tau^{(k-k_0)/(n+\alpha)} \sqrt{n} r_i^k > \sqrt{n} r_i^k$ . Let  $q = p(1 + \frac{\alpha}{n})$  for simplicity. Then for any  $n/(n+\alpha) < p \leq 1$ ,  $w \in A_q$  and  $q > 1$ , we can see that  $[n(q_w/p - 1)] = 0$ . Obviously, by Theorem 2.3, we know that all the functions  $b_i^k$  satisfy the conditions in Lemma 3.1. Applying Lemma 2.2 and Lemma 3.1, we can deduce

$$\begin{aligned} I_2'' &\leq C \sum_{k=k_0+1}^{\infty} \sum_i 2^{kp} (r_i^k)^{(n+\alpha)p} \int_{|x-x_i^k| \geq \tau^{(k-k_0)/(n+\alpha)} \sqrt{n} r_i^k} \frac{w(x)}{|x - x_i^k|^{(n+\alpha)p}} dx \\ &= C \sum_{k=k_0+1}^{\infty} \sum_i 2^{kp} (r_i^k)^{nq} \int_{|y| \geq \tau^{(k-k_0)/(n+\alpha)} \sqrt{n} r_i^k} \frac{w_1(y)}{|y|^{nq}} dy \\ &\leq C \sum_{k=k_0+1}^{\infty} \sum_i 2^{kp} (\tau^{(k-k_0)/(n+\alpha)})^{-nq} w_1(Q(0, \tau^{(k-k_0)/(n+\alpha)} \cdot r_i^k)) \\ &= C \sum_{k=k_0+1}^{\infty} \sum_i 2^{kp} (\tau^{(k-k_0)/(n+\alpha)})^{-nq} w(Q(x_i^k, \tau^{(k-k_0)/(n+\alpha)} \cdot r_i^k)), \end{aligned}$$

where  $w_1(x) = w(x + x_i^k)$  is the translation of  $w(x)$ . It is obvious that  $w_1 \in A_q$  whenever  $w \in A_q$ , and  $q_{w_1} = q_w$ . In addition, for  $w \in A_q$  with  $q > 1$ , then we can take a sufficiently small number  $\varepsilon > 0$  such that  $w \in A_{q-\varepsilon}$ . Therefore, by using Lemma 2.1 again, we obtain

$$\begin{aligned} I_2'' &\leq C \sum_{k=k_0+1}^{\infty} \sum_i 2^{kp} (\tau^{(k-k_0)/(n+\alpha)})^{-n\varepsilon} w(Q_i^k) \\ &\leq C \|f\|_{WH_w^p}^p \sum_{k=k_0+1}^{\infty} (\tau^{(k-k_0)/(n+\alpha)})^{-n\varepsilon} \\ &\leq C \|f\|_{WH_w^p}^p. \end{aligned}$$

Summing up the above estimates for  $I_1$  and  $I_2$  and then taking the supremum over all  $\lambda > 0$ , we complete the proof of Theorem 1.1.  $\square$

## 4 Proof of Theorem 1.2

In order to prove Theorem 1.2, we shall need the following two lemmas.

**Lemma 4.1.** *Let  $0 < \alpha \leq 1$ ,  $n/(n + \alpha) < p \leq 1$  and  $w \in A_{p(1 + \frac{\alpha}{n})}$ . Then for every  $\lambda > p(1 + \frac{\alpha}{n})$ , we have*

$$\|g_{\lambda, \alpha}^*(f)\|_{L_w^2} \leq C\|f\|_{L_w^2}$$

holds for all  $f \in L_w^2(\mathbb{R}^n)$ .

*Proof.* From the definition, we readily see that

$$\begin{aligned} (g_{\lambda, \alpha}^*(f)(x))^2 &= \iint_{\mathbb{R}_+^{n+1}} \left( \frac{t}{t + |x - y|} \right)^{\lambda n} (A_\alpha(f)(y, t))^2 \frac{dy dt}{t^{n+1}} \\ &= \int_0^\infty \int_{|x-y| < t} \left( \frac{t}{t + |x - y|} \right)^{\lambda n} (A_\alpha(f)(y, t))^2 \frac{dy dt}{t^{n+1}} \\ &\quad + \sum_{j=1}^\infty \int_0^\infty \int_{2^{j-1}t \leq |x-y| < 2^j t} \left( \frac{t}{t + |x - y|} \right)^{\lambda n} (A_\alpha(f)(y, t))^2 \frac{dy dt}{t^{n+1}} \\ &\leq C \left[ \mathcal{S}_\alpha(f)(x)^2 + \sum_{j=1}^\infty 2^{-j\lambda n} \mathcal{S}_{\alpha, 2^j}(f)(x)^2 \right]. \end{aligned} \quad (4)$$

We are now going to estimate  $\int_{\mathbb{R}^n} |\mathcal{S}_{\alpha, 2^j}(f)(x)|^2 w(x) dx$  for  $j = 1, 2, \dots$ . Fubini's theorem and Lemma 2.1 imply

$$\begin{aligned} \int_{\mathbb{R}^n} |\mathcal{S}_{\alpha, 2^j}(f)(x)|^2 w(x) dx &= \iint_{\mathbb{R}_+^{n+1}} \left( \int_{|x-y| < 2^j t} w(x) dx \right) (A_\alpha(f)(y, t))^2 \frac{dy dt}{t^{n+1}} \\ &\leq C \cdot 2^{j(n+\alpha)p} \iint_{\mathbb{R}_+^{n+1}} \left( \int_{|x-y| < t} w(x) dx \right) (A_\alpha(f)(y, t))^2 \frac{dy dt}{t^{n+1}} \\ &= C \cdot 2^{j(n+\alpha)p} \|\mathcal{S}_\alpha(f)\|_{L_w^2}^2. \end{aligned} \quad (5)$$

Since  $w \in A_{p(1 + \frac{\alpha}{n})}$  and  $1 < p(1 + \frac{\alpha}{n}) \leq 2$ , then we have  $w \in A_2$ . Therefore, under the assumption that  $\lambda > p(1 + \frac{\alpha}{n})$ , it follows from Theorem A and the above inequalities (4) and (5) that

$$\begin{aligned} \|g_{\lambda, \alpha}^*(f)\|_{L_w^2}^2 &\leq C \left( \int_{\mathbb{R}^n} |\mathcal{S}_\alpha(f)(x)|^2 w(x) dx + \sum_{j=1}^\infty 2^{-j\lambda n} \int_{\mathbb{R}^n} |\mathcal{S}_{\alpha, 2^j}(f)(x)|^2 w(x) dx \right) \\ &\leq C \left( \|\mathcal{S}_\alpha(f)\|_{L_w^2}^2 + \sum_{j=1}^\infty 2^{-j\lambda n} \cdot 2^{j(n+\alpha)p} \|\mathcal{S}_\alpha(f)\|_{L_w^2}^2 \right) \\ &\leq C \cdot \|f\|_{L_w^2}^2 \left( 1 + \sum_{j=1}^\infty 2^{-j\lambda n} \cdot 2^{j(n+\alpha)p} \right) \\ &\leq C \cdot \|f\|_{L_w^2}^2. \end{aligned}$$

We are done.  $\square$

Following the same procedure as that of Lemma 3.1, we can also show

**Lemma 4.2.** *Let  $0 < \alpha \leq 1$  and  $j \in \mathbb{Z}_+$ . Then for any given function  $b \in L^\infty(\mathbb{R}^n)$  with support contained in  $Q = Q(x_0, r)$ , and  $\int_{\mathbb{R}^n} b(x) dx = 0$ , we have*

$$\mathcal{S}_{\alpha, 2^j}(b)(x) \leq C \cdot 2^{j(3n+2\alpha)/2} \|b\|_{L^\infty} \frac{r^{n+\alpha}}{|x - x_0|^{n+\alpha}}, \quad \text{whenever } |x - x_0| > \sqrt{n}r.$$

*Proof.* For any  $z \in Q(x_0, r)$ , we have  $|z - x_0| < \frac{|x - x_0|}{2}$ . Then for all  $(y, t) \in \Gamma_{2^j}(x)$  and  $|z - y| \leq t$  with  $z \in Q$ , as in the proof of Lemma 3.1, we can deduce that

$$t + 2^j t \geq |x - y| + |y - z| \geq |x - z| \geq |x - x_0| - |z - x_0| \geq \frac{|x - x_0|}{2}. \quad (6)$$

Thus, by using the inequalities (1) and (6), we obtain

$$\begin{aligned} |\mathcal{S}_{\alpha, 2^j}(b)(x)| &= \left( \iint_{\Gamma_{2^j}(x)} \left( \sup_{\varphi \in \mathcal{C}_\alpha} |(\varphi_t * b)(y)| \right)^2 \frac{dy dt}{t^{n+1}} \right)^{1/2} \\ &\leq C \cdot \|b\|_{L^\infty} r^{n+\alpha} \left( \int_{\frac{|x-x_0|}{2^{j+2}}}^\infty \int_{|y-x| < 2^j t} \frac{dy dt}{t^{2(n+\alpha)+n+1}} \right)^{1/2} \\ &\leq C \cdot 2^{jn/2} \|b\|_{L^\infty} r^{n+\alpha} \left( \int_{\frac{|x-x_0|}{2^{j+2}}}^\infty \frac{dt}{t^{2(n+\alpha)+1}} \right)^{1/2} \\ &\leq C \cdot 2^{j(3n+2\alpha)/2} \|b\|_{L^\infty} \frac{r^{n+\alpha}}{|x - x_0|^{n+\alpha}}. \end{aligned}$$

This finishes the proof of the lemma.  $\square$

We are ready to show our main theorem of this section.

*Proof of Theorem 1.2.* We follow the strategy of the proof of Theorem 1.1. For any given  $\sigma > 0$ , we are able to choose  $k_0 \in \mathbb{Z}$  such that  $2^{k_0} \leq \sigma < 2^{k_0+1}$ . For every  $f \in WH_w^p(\mathbb{R}^n)$ , we can also write

$$\begin{aligned} &\sigma^p \cdot w(\{x \in \mathbb{R}^n : |g_{\lambda, \alpha}^*(f)(x)| > \sigma\}) \\ &\leq \sigma^p \cdot w(\{x \in \mathbb{R}^n : |g_{\lambda, \alpha}^*(F_1)(x)| > \sigma/2\}) + \sigma^p \cdot w(\{x \in \mathbb{R}^n : |g_{\lambda, \alpha}^*(F_2)(x)| > \sigma/2\}) \\ &= J_1 + J_2, \end{aligned}$$

where the notations  $F_1$  and  $F_2$  are the same as in the proof of Theorem 1.1. By our assumption, we know that  $\lambda > (3n + 2\alpha)/n > p(1 + \frac{\alpha}{n})$ . Applying Chebyshev's inequality, Lemma 4.1 and the previous inequality (3), we obtain

$$\begin{aligned} J_1 &\leq \sigma^p \cdot \frac{4}{\sigma^2} \|g_{\lambda, \alpha}^*(F_1)\|_{L_w^2}^2 \\ &\leq C \cdot \sigma^{p-2} \|F_1\|_{L_w^2}^2 \\ &\leq C \|f\|_{WH_w^p}^p. \end{aligned}$$

To estimate the other term  $J_2$ , as before, we also set

$$A_{k_0} = \bigcup_{k=k_0+1}^{\infty} \bigcup_i \widetilde{Q}_i^k,$$

where  $\widetilde{Q}_i^k = Q(x_i^k, \tau^{(k-k_0)/(n+\alpha)}(2\sqrt{n})r_i^k)$ ,  $\tau$  is also a fixed real number such that  $1 < \tau < 2$  and  $\text{supp } b_i^k \subseteq Q_i^k = Q(x_i^k, r_i^k)$ . Again, we shall further decompose  $J_2$  as

$$\begin{aligned} J_2 &\leq \sigma^p \cdot w(\{x \in A_{k_0} : |g_{\lambda, \alpha}^*(F_2)(x)| > \sigma/2\}) + \sigma^p \cdot w(\{x \in (A_{k_0})^c : |g_{\lambda, \alpha}^*(F_2)(x)| > \sigma/2\}) \\ &= J'_2 + J''_2. \end{aligned}$$

Using the same arguments as that of Theorem 1.1, we can see that

$$\begin{aligned} J'_2 &\leq \sigma^p \sum_{k=k_0+1}^{\infty} \sum_i w(\widetilde{Q}_i^k) \\ &\leq C \cdot \sigma^p \sum_{k=k_0+1}^{\infty} \tau^{(k-k_0)p} \sum_i w(Q_i^k) \\ &\leq C \|f\|_{WH_w^p}^p. \end{aligned}$$

Noting that  $0 < p \leq 1$ . Then by Chebyshev's inequality and the inequality (4), we have

$$\begin{aligned} J''_2 &\leq 2^p \int_{(A_{k_0})^c} |g_{\lambda, \alpha}^*(F_2)(x)|^p w(x) dx \\ &\leq 2^p \sum_{k=k_0+1}^{\infty} \sum_i \int_{(\widetilde{Q}_i^k)^c} |S_{\alpha}(b_i^k)(x)|^p w(x) dx \\ &\quad + 2^p \sum_{j=1}^{\infty} 2^{-j\lambda np/2} \sum_{k=k_0+1}^{\infty} \sum_i \int_{(\widetilde{Q}_i^k)^c} |S_{\alpha, 2^j}(b_i^k)(x)|^p w(x) dx \\ &= K_0 + \sum_{j=1}^{\infty} 2^{-j\lambda np/2} K_j. \end{aligned}$$

Note that  $[n(q_w/p - 1)] = 0$  by the hypothesis. Clearly, in view of Theorem 2.3, we can easily see that all the functions  $b_i^k$  satisfy the conditions in Lemma 3.1 or Lemma 4.2. In the proof of Theorem 1.1, we have already showed that

$$K_0 \leq C \|f\|_{WH_w^p}^p.$$

Below we shall give the estimates of  $K_j$  for every  $j = 1, 2, \dots$ . Observe that for any  $x \in (\widetilde{Q}_i^k)^c$ , we have  $|x - x_i^k| \geq \tau^{(k-k_0)/(n+\alpha)} \sqrt{n} r_i^k > \sqrt{n} r_i^k$ . Since  $\|b_i^k\|_{L^\infty} \leq C 2^k$ , then, using Lemma 4.2 and following the same lines as in

Theorem 1.1, we can also deduce

$$\begin{aligned}
K_j &\leq C \cdot 2^{j(3n+2\alpha)p/2} \sum_{k=k_0+1}^{\infty} \sum_i \|b_i^k\|_{L^\infty}^p (r_i^k)^{(n+\alpha)p} \int_{|x-x_i^k| \geq \tau^{(k-k_0)/(n+\alpha)} \sqrt{n} r_i^k} \frac{w(x)}{|x-x_i^k|^{(n+\alpha)p}} dx \\
&\leq C \cdot 2^{j(3n+2\alpha)p/2} \sum_{k=k_0+1}^{\infty} \sum_i 2^{kp} (r_i^k)^{(n+\alpha)p} \int_{|x-x_i^k| \geq \tau^{(k-k_0)/(n+\alpha)} \sqrt{n} r_i^k} \frac{w(x)}{|x-x_i^k|^{(n+\alpha)p}} dx \\
&\leq C \cdot 2^{j(3n+2\alpha)p/2} \|f\|_{WH_w^p}^p.
\end{aligned}$$

Hence, we finally obtain

$$\begin{aligned}
J_2'' &\leq C \|f\|_{WH_w^p}^p \left( 1 + \sum_{j=1}^{\infty} 2^{-j\lambda np/2} \cdot 2^{j(3n+2\alpha)p/2} \right) \\
&\leq C \|f\|_{WH_w^p}^p,
\end{aligned}$$

where the last series is convergent since  $\lambda > (3n+2\alpha)/n$ . Therefore, combining the above estimates for  $J_1$  and  $J_2$  and then taking the supremum over all  $\sigma > 0$ , we conclude the proof of Theorem 1.2.  $\square$

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